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LETTER TO THE EDITOR

Noise effects on synchronization in systems of coupled oscillators

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Abstract. We study the synchronization phenomena in systems of globally coupled oscillators, each possessing finite inertia, with particular attention to the noise effects. The self-consistency equation for the order parameter as well as the probability distribution is obtained from the Smoluchowski equation, and analyzed in the presence of thermal noise. It is found that the hysteresis present in the system without noise disappears as the thermal noise comes into the system. Numerical simulations are also performed to give results generally consistent with the analytical ones.

Due to its many applications and prevalence in the systems of physics, chemistry, biology, and the social sciences, one of the remarkable features of a set of coupled oscillators, *collective synchronization*, has attracted much interest [1–4]. Collective synchronization appears in a variety of self-organizing systems, which may be modelled by sets of coupled nonlinear oscillators [5–7]. The system of globally coupled oscillators has analytic simplicity and some physical as well as biological applications, and has been most studied, both analytically and numerically. The effects of nonzero inertia of each oscillator in such a system of globally coupled oscillators have also been examined, and appearance of hysteresis together with discontinuous transitions between the coherent and incoherent states have been pointed out [8, 9]. Such hysteresis due to the inertia is well known in the response of a single oscillator (without regard to synchronization). In this simple case, it has also been observed numerically that the presence of noise strongly suppresses the hysteresis [10].

The purpose of this paper is to investigate the effects of the thermal noise on collective synchronization in the system of coupled oscillator with finite inertia. In particular, how the thermal noise affects the hysteretic behaviour of the system is examined in an analytical way. For this purpose, we employ the Smoluchowski equation with the correction due to the inertia [11], derive the self-consistency equation for the order parameter as well as the probability distribution, and investigate the resulting collective synchronization phenomena in the presence of the noise. It is found that the hysteresis in the synchronization is strongly suppressed and disappears as the thermal noise comes into the system. We also present results of numerical simulations, which indeed demonstrate strong suppression of the hysteresis by the noise.

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We begin with the equations of motion for a set of N oscillators, the ith of which is described by its phase ϕ_i $(i = 1, 2, \ldots, N)$:

$$\mu\ddot{\phi}_i + \dot{\phi}_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\phi_i - \phi_j) = \omega_i + \Gamma_i(t) \tag{1}$$

where μ denotes the magnitude of the inertia relative to the damping. The third term on the left-hand side represents the global coupling between oscillators, with strength K/N. The first and the second terms on the right-hand side describe the constant driving and the random (thermal) noise, respectively. The constant driving strength ω_i is distributed over the whole oscillators according to the distribution $g(\omega)$, which is assumed to be smooth and symmetric about ω_0. Without loss of generality, we may take ω_0 to be zero and also assume that $g(\omega)$ is concave at $\omega = 0$. The noise $\Gamma_i(t)$'s are independent white noise with zero mean and correlation $\langle\Gamma_i(t)\Gamma_j(t')\rangle = 2T\delta_{ij}\delta(t - t')$, where $T(> 0)$ may be regarded as the effective temperature of the system. Collective synchronization of such an N oscillator system is conveniently described by the complex order parameter

$$\Psi \equiv \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}\phi_j} = \Delta\mathrm{e}^{\mathrm{i}\theta} \tag{2}$$

where Δ indicates the magnitude of the order parameter and θ denotes the average phase. Here, nonvanishing Ψ indicates the occurrence of collective synchronization. The order parameter defined in equation (2) allows us to reduce equation (1) to a *single* decoupled equation

$$\mu\ddot{\phi}_i + \dot{\phi}_i + K\Delta\sin(\phi_i - \theta) = \omega_i + \Gamma_i(t) \tag{3}$$

where Δ and θ are to be determined by imposing self-consistency. Redefining $\phi_i - \theta$ as ϕ_i and suppressing indices for simplicity, we obtain

$$\mu\ddot{\phi} + \dot{\phi} + K\Delta\sin\phi = \omega + \Gamma(t). \tag{4}$$

It is convenient to consider the corresponding Fokker–Planck equation for the probability distribution $P(\phi, \dot{\phi}, t)$ rather than the Langevin equation in equation (4). Further, taking the average over the velocity $\dot{\phi}$ in the long-time limit reduces the Fokker–Planck equation into the Smoluchowski equation, which reads [11]

$$\frac{\partial P(\phi, t)}{\partial t} = \frac{\partial}{\partial\phi}\left[\left(\frac{\partial V(\phi)}{\partial\phi}P(\phi) + T\frac{\partial P(\phi)}{\partial\phi}\right)\left(1 + \mu\frac{\partial^2 V(\phi)}{\partial\phi^2}\right)\right] \tag{5}$$

with the washboard potential $V(\phi) \equiv -K\Delta\cos\phi - \omega\phi$. Note the correction term due to the finite inertia μ in equation (5). In the stationary state, the left-hand side of equation (5) vanishes, leading to

$$(\omega - K\Delta\sin\phi)P(\phi) - T\frac{\partial P(\phi)}{\partial\phi} = SF(\phi) \tag{6}$$

where S is a constant and $F(\phi) \equiv [1 + \mu(\partial^2 V/\partial\phi^2)]^{-1} = (1 + \mu K\Delta\cos\phi)^{-1}$. At zero (effective) temperature ($T = 0$), i.e. in the absence of noise, it is straightforward to obtain the stationary probability distribution [9]

$$P(\phi) = \begin{cases} \mathcal{N}|\omega - K\Delta\sin\phi|^{-1}(1 + \mu K\Delta\cos\phi)^{-1} & \text{for } |\omega| > K\Delta \\ \delta[\phi - \sin^{-1}(\omega/K\Delta)] & \text{for } |\omega| \leqslant K\Delta \end{cases} \tag{7}$$

where $\mathcal{N}$ is the normalization constant determined by the relation $\int_0^{2\pi}\mathrm{d}\phi\, P(\phi) = 1$.

In the presence of noise ($T \neq 0$), the solution of equation (6) yields the probability distribution

$$P(\phi) = N\mathrm{e}^{-V(\phi)/T}\left[1 - \frac{S}{NT}\int_0^{\phi} F(\phi')\mathrm{e}^{V(\phi')/T}\,\mathrm{d}\phi'\right]$$
$$= N\mathrm{e}^{-V(\phi)/T}\left[1 - \frac{1-\mathrm{e}^{-2\pi\omega/T}}{\int_0^{2\pi}\mathrm{d}\phi'\,F(\phi')\mathrm{e}^{V(\phi')/T}}\int_0^{\phi}\mathrm{d}\phi''\,F(\phi'')\mathrm{e}^{V(\phi'')/T}\right] \quad (8)$$

where the 2π-periodic condition $P(\phi + 2\pi) = P(\phi)$ has been used to obtain S, and the normalization condition determines N:

$$N = \left\{\int_0^{2\pi}\mathrm{d}\phi\,\mathrm{e}^{-V(\phi)/T}\left[1 - \frac{1-\mathrm{e}^{-2\pi\omega/T}}{\int_0^{2\pi} F(\phi')\mathrm{e}^{V(\phi')/T}\,\mathrm{d}\phi'}\int_0^{\phi}\mathrm{d}\phi''\,F(\phi'')\mathrm{e}^{V(\phi'')/T}\right]\right\}^{-1}.$$

Since we are interested in the transition between the incoherent state ($\Delta = 0$) and the coherent one ($\Delta \neq 0$), it is natural to assume that $\mu K\Delta \ll 1$ near the transition, which allows the expansion: $F(\phi) = (1+\mu K\Delta\cos\phi)^{-1} = 1-\mu K\Delta\cos\phi+(\mu K\Delta)^2\cos^2\phi-(\mu K\Delta)^3\cos^3\phi+\mathrm{O}(K\Delta)^4$. We further use the series expansion of $\mathrm{e}^{x\cos\phi}$ in terms of the modified Bessel functions: $\mathrm{e}^{x\cos\phi} = I_0(x) + 2\sum_{n=1}^{\infty} I_n(x)\cos n\phi$. Putting these expansions and performing integration, we write equation (8) in the form

$$P(\phi) = Z^{-1} f(\phi)\mathrm{e}^{(K\Delta/T)\cos\phi} \quad (9)$$

with

$$\begin{aligned}
f(\phi) \equiv{}& I_0(x) - 2\sum_{n=1}^{\infty}(-1)^n I_n(x) Q_n(y;\phi)\\
&+\mu K\Delta\left\{I_0(x)Q_1(y;\phi) + \sum_{n=1}^{\infty}(-1)^n I_n(x)[Q_{n+1}(y;\phi) + Q_{n-1}(y;\phi)]\right\}\\
&+(\mu K\Delta)^2\left\{\tfrac{1}{2}I_0(x) - \tfrac{1}{2}I_0(x)Q_2(y;\phi) - \sum_{n=1}^{\infty}(-1)^n Q_n(y;\phi)\right.\\
&\left.-\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n I_n(x)[Q_{n+2}(y;\phi) + Q_{n-2}(y;\phi)]\right\}\\
&+(\mu K\Delta)^3\left\{\tfrac{3}{4}I_0(x)Q_1(y;\phi) + \tfrac{3}{4}\sum_{n=1}^{\infty}(-1)^n I_n(x)[Q_{n+1}(y;\phi) + Q_{n-1}(y;\phi)]\right.\\
&\left.+\tfrac{1}{4}I_0(x)Q_3(y;\phi) + \tfrac{1}{4}\sum_{n=1}^{\infty}(-1)^n I_n(x)[Q_{n+3}(y;\phi) + Q_{n-3}(y;\phi)]\right\}\\
&+\mathrm{O}(\mu K\Delta)^4
\end{aligned} \quad (10)$$

and

$$\begin{aligned}
Z \equiv{}& 2\pi I_0^2(x) + 4\pi y^2\sum_{n=1}^{\infty}(-1)^n I_n^2(x)R_n(y) - \mu K\Delta\left\{2\pi I_0(x)I_1(x)y^2R_1(y)\right.\\
&\left.+2\pi y^2\sum_{n=1}^{\infty}(-1)^n I_n(x)[I_{n+1}(x)R_{n+1}(y) + I_{n-1}(x)R_{n-1}(y)]\right\}\\
&+(\mu K\Delta)^2\left\{\pi I_0^2(x) + \pi I_0(x)I_2(x)y^2R_2(y) + 2\pi y^2\sum_{n=1}^{\infty}(-1)^n I_n^2(x)R_n(y)\right.\\
&\left.+\pi y^2\sum_{n=1}^{\infty}(-1)^n I_n(x)[I_{n+2}(x)R_{n+2}(y) + I_{n-2}(x)R_{n-2}(y)]\right\}
\end{aligned}$$

$$-(\mu K\Delta)^3\Bigg\{\tfrac{3}{2}\pi I_0(x)I_1(x)y^2R_1(y)+\tfrac{3}{2}\pi y^2\sum_{n=1}^{\infty}(-1)^nI_n(x)$$
$$\times[I_{n+1}(x)R_{n+1}(y)+I_{n-1}(x)R_{n-1}(y)]+\frac{\pi}{2}I_0(x)I_3(x)y^2R_3(y)$$
$$+\frac{\pi}{2}y^2\sum_{n=1}^{\infty}(-1)^nI_n(x)[I_{n+3}(x)R_{n+3}(y)+I_{n-3}(x)R_{n-3}(y)]\Bigg\}+\mathrm{O}(\mu K\Delta)^4 \quad (11)$$

where $x \equiv K\Delta/T$, $y \equiv \omega/T$, $Q_n(y;\phi) \equiv (n^2+y^2)^{-1}(ny\sin n\phi - y^2\cos n\phi)$, and $R_n(y) \equiv (n^2+y^2)^{-1}$.

We now derive the self-consistency equation for the order parameter, which determines the collective behaviour of the system. Recalling that ϕ in equation (4) in fact represents $\phi-\theta$, we have the self-consistency equation

$$\Delta = \frac{1}{N}\sum_j \mathrm{e}^{\mathrm{i}\phi_j} = \int_{-\infty}^{\infty}\mathrm{d}\omega\, g(\omega)\langle \mathrm{e}^{\mathrm{i}\phi}\rangle_\omega \quad (12)$$

where $\langle\cdots\rangle_\omega$ denotes the average in the stationary state with given ω.

At zero temperature the average in equation (12) is taken with respect to the probability distribution given by equation (7), leading to the order parameter [9]

$$\Delta = \left(\frac{\pi}{2}-\frac{\mu}{2}\right)g(0)K\Delta + \frac{4}{3}\mu g(0)(K\Delta)^2 + \frac{\pi}{16}g''(0)(K\Delta)^3 + \mathrm{O}(K\Delta)^4. \quad (13)$$

Here in the presence of the inertia ($\mu \neq 0$), unlocked oscillators as well as those locked to the external (constant) driving contribute to the collective synchronization, and the resulting quadratic term of the order $(K\Delta)^2$ is expected to induce hysteresis in the bifurcation diagram [12]. The appearance of such hysteresis at zero temperature has been pointed out in the system with non-vanishing inertia [8, 9].

We then take into account the effects of noise, which are described by the stationary distribution given by equation (9). Again, with the expectation that $K\Delta$ is small near the transition, we consider for simplicity the range of the temperature such as $K\Delta/T \ll 1$. In this case, where the noise strength is not so weak, the Bessel functions in equations (10) and (11) can be expanded, and the corresponding stationary distribution employed in equation (12) leads to the self-consistency equation for the order parameter:

$$\Delta = a(K\Delta) - c(K\Delta)^3 + \mathrm{O}(K\Delta)^5 \quad (14)$$

with the coefficients given by the integrals

$$a = \int_{-\infty}^{\infty}\mathrm{d}\omega\, g(\omega)\frac{T-\mu\omega^2}{2(T^2+\omega^2)}$$
$$c = \int_{-\infty}^{\infty}\mathrm{d}\omega\, g(\omega)\Bigg[\frac{T+\mu(T^2-\omega^2)-\mu^2\omega^2T}{4(T^2+\omega^2)^2}+\frac{\mu^3\omega^2+2\mu^2T}{8(T^2+\omega^2)}$$
$$-\frac{6T+\mu(8T^2-\omega^2)+\mu^2T(8T^2-\omega^2)}{8(T^2+\omega^2)(4T^2+\omega^2)}\Bigg].$$

The above integrals can be evaluated when the distribution of the constant driving ω is given. For example, in the case of a Gaussian distribution, the above coefficients a and c can be computed in terms of the error function. Note that the above expressions reproduce the known forms [5] in the absence of the inertia and noise ($\mu = 0$ and $T \to 0$): $a = (\pi/2)g(0)$ and $c = -(\pi/16)g''(0)$.

The collective behaviour of the system for given values of a and c can be obtained by solving equation (14). When $K < K_c \equiv 1/a$, only the null solution ($\Delta = 0$) is possible. At $K = K_c$, on the other hand, the null solution loses its stability and the nontrivial solution

$$\Delta = \Delta_+ \equiv \frac{\sqrt{cK(aK-1)}}{cK^2} \tag{15}$$

together with the unphysical solution $\Delta_- \equiv -\Delta_+$, emerges via a pitchfork bifurcation. Subsequently, it grows in a continuous manner $(a^2/\sqrt{c})(K-K_c)^{1/2}$ as K is increased beyond K_c [12]. It is of particular interest here that the quadratic term of the order $(K\Delta)^2$, which gives rise to hysteresis in the zero-temperature system described by equation (13), does not appear in equation (14). It is thus concluded that the hysteresis present in the system without noise is suppressed as the noise comes into the system. Such suppression of hysteresis due to noise has also been observed numerically in a single oscillator, corresponding to equation (4) with the coupling $K\Delta$ fixed [10].

To confirm the analytical results presented above, we have performed numerical simulations with the equations of motion given by equation (1) at various noise and coupling strengths. The order parameter Δ has been computed according to the definition given by equation (2), and its behaviour with the coupling strength has been examined. For convenience, a semicircle distribution of radius 0.5 has been employed for $g(\omega)$, and equation (1) has been integrated with discrete time steps of $\Delta t = 0.001$. (The results do not change qualitatively even if other distribution such as a Gaussian is used.) At each run, we have used $N_t = 10^5$ time steps to compute the order parameter, discarding the data from the first 5×10^4 steps, and varied both Δt and N_t to verify that the stationary state was achieved. Finally, independent runs with 20 different distributions of the constant driving strength and initial conditions have been performed, over which the averages have been taken.

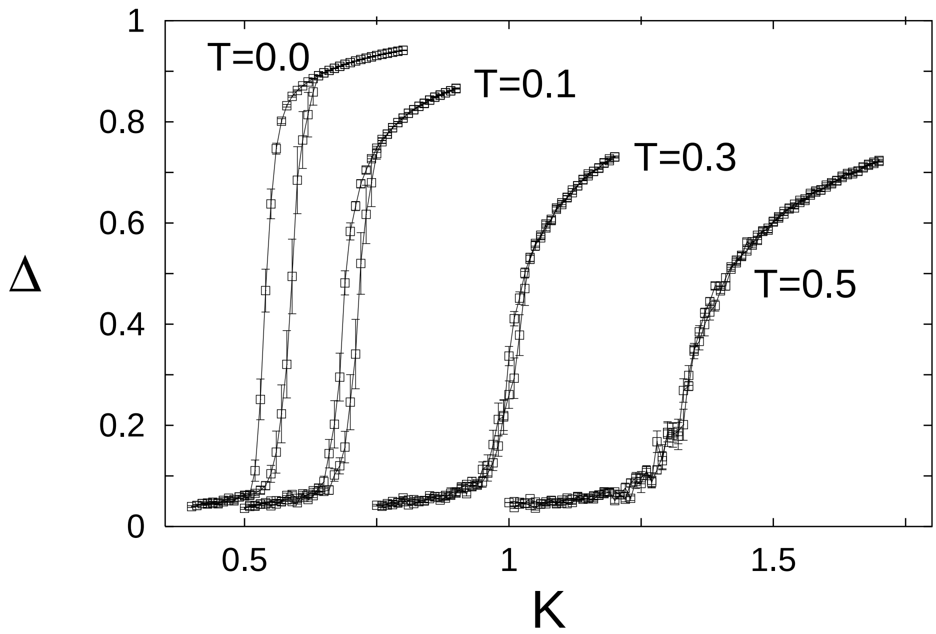

Figure 1. Behaviour of the order parameter with the coupling strength, at various noise strengths. Error bars represent standard deviations, and lines are merely guides to the eye.

We have thus computed the order parameter in the system of N oscillators, for N up to 2000, and confirmed that there is no appreciable finite-size effects for $N \gtrsim 1000$. Figure 1 presents the resulting behaviour of the order parameter at various noise strengths, in a system of $N = 2000$ oscillators, each having the inertia $\mu = 0.8$: hysteresis is indeed manifested at zero noise and observed to weaken strikingly, as the noise strength is increased. We have also considered other values of the inertia μ to find largely similar behaviour except for the increase of hysteresis with μ, as expected. It is thus concluded that noise not only hinders synchronization, making the critical coupling strength K_c larger, but also strongly suppresses hysteresis coming from the non-vanishing inertia.

In conclusion, we have studied both analytically and numerically the synchronization phenomena in a set of globally coupled oscillators, each possessing finite inertia, with particular attention to the noise effects. The synchronization phenomena in the system without noise are characterized by bistability and associated hysteresis, resulting from the inertia. Here it has been found analytically that noise crucially changes the synchronization phenomena, strongly suppressing such hysteresis: In the presence of noise, the system displays only the continuous transition between the coherent and incoherent states, and no hysteretic behaviour can be observed. Namely, the order parameter grows continuously from zero, which indicates that $K\Delta$ becomes vanishingly small near the transition. Consequently, the condition $K\Delta/T \ll 1$, under which the self-consistency equation for the order parameter Δ (equation (14)) has been derived, is indeed satisfied near the transition, making the analysis consistent. However, it should be noted that the condition may not hold in the zero-noise limit ($T \to 0$), making it unclear whether even arbitrarily small noise destroys the hysteretic behaviour completely. Indeed numerical simulations seem to favour the existence of a finite critical noise strength, below which hysteresis still persists.

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References

[1] Winfree A T 1980 *The Geometry of Biological Time* (New York: Springer)
Kuramoto Y 1984 *Chemical Oscillations, Waves, and Turbulence* (Berlin: Springer)
[2] Benz S P, Rzchowski M, Tinkham M and Lobb C J 1990 *Phys. Rev. Lett.* **64** 693
Choi M Y 1992 *Phys. Rev.* B **46** 564
Kim S, Kim B J and Choi M Y 1995 *Phys. Rev.* B **52** 13 536
Park K and Choi M Y 1997 *Phys. Rev.* B **56** 387
[3] Walker T J 1969 *Science* **166** 891
McClintock M K 1971 *Nature* **229** 244
Michaels D C, Maytyas E P and Jalife J 1987 *Circ. Res.* **61** 704
Buck J 1988 *Q. Rev. Biol.* **63** 265
Traub R D, Miles R and Wong R K S 1989 *Science* **243** 1319
[4] Eckhorn R, Bauer R, Jordan W, Brosch M, Kruse W, Munk M and Reitboeck H J 1988 *Biol. Cybern.* **60** 121
Gray C M and Singer W 1989 *Proc. Natl Sci., USA* **86** 1698
Park K and Choi M Y 1995 *Phys. Rev.* E **52** 2907
[5] Kuramoto Y and Nishikawa I 1987 *J. Stat. Phys.* **49** 569
[6] Daido H 1987 *Prog. Theor. Phys.* **77** 622
Daido H 1992 *Phys. Rev. Lett.* **68** 1073
[7] Swift J W, Strogatz S H and Wiesenfeld K 1992 *Physica* D **55** 239
Strogatz S H, Mirollo R E and Matthews P C 1992 *Phys. Rev. Lett.* **68** 2730
Watanabe S and Strogatz S H 1993 *Phys. Rev. Lett.* **70** 2391
Wiesenfeld K and Swift J W 1995 *Phys. Rev.* E **51** 1020

Wiesenfeld K, Colet P and Strogatz S H 1996 *Phys. Rev. Lett.* **76** 404
[8] Tanaka H, Lichtenberg A J and Oishi S 1997 *Phys. Rev. Lett.* **78** 2104
[9] Hong H, Choi M Y, Yi J and Soh K-S 1999 *Phys. Rev.* E **59** at press
[10] Risken H 1989 *The Fokker–Planck Equation: Methods of Solution and Applications* (Berlin: Springer) pp 328–37
[11] Schneider T, Stoll E P and Morf R 1978 *Phys. Rev.* B **18** 1417
[12] Choi M Y, Kim Y W and Hong D C 1994 *Phys. Rev.* E **49** 3825